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Compactifications with S-duality twists

Christopher M. Hull

Department of Physics Queen Mary, University of London
Mile End Rd, London E1 4NS, U.K.
E-mail: c.m.hull@qmul.ac.uk

Aybike Çatal-Özer

Department of Physics, Middle East Technical University
İnönü Bulvarı Yolu, 06531, Ankara, Turkey
E-mail: catal@metu.edu.tr

Abstract: We consider generalised Scherk Schwarz reductions of supergravity and superstring theories with twists by electromagnetic dualities that are symmetries of the equations of motion but not of the action, such as the S-duality of $D = 4$, $N = 4$ super-Yang-Mills coupled to supergravity. The reduction cannot be done on the action itself, but must be done either on the field equations or on a duality invariant form of the action, such as one in the doubled formalism in which potentials are introduced for both electric and magnetic fields. The resulting theory in odd-dimensions has massive form fields satisfying a self-duality condition $dA \sim m * A$. We construct such theories in $D = 3, 5, 7$.

Keywords: Superstring Vacua, Supergravity Models, String Duality.

Contents

1. Introduction

Twisted toroidal compactifications or Scherk-Schwarz reductions are a useful way of introducing masses into supergravity and string compactifications, generating a potential for the scalar fields [1]–[19]. A theory in $D+1$ dimensions with global symmetry G can be compactified on a circle with fields not periodic but with a G monodromy around the circle, and the monodromy introduces masses into the theory and breaks some of the symmetry. The purpose here is to generalise such compactifications to the case in which G is a symmetry of the equations of motion only, not of the action; we shall refer to such symmetries here as S-dualities. A standard example is S-duality in 4-dimensions. The heterotic string compactified to four dimensions has a classical $\mathrm{SL}(2,\mathbb{R})$ symmetry which acts through electromagnetic duality transformations and so is only a symmetry of the equations of motion. In this case, we consider a circle reduction to three dimensions with a monodromy in $\mathrm{SL}(2,\mathbb{R})$. In the quantum theory, the $\mathrm{SL}(2,\mathbb{R})$ symmetry is broken to $\mathrm{SL}(2,\mathbb{Z})$ [20] and in that case the monodromy must be in $\mathrm{SL}(2,\mathbb{Z})$ [6]. We generalise this to other dimensions, and discuss examples in $D = 3, 5$ and 7 dimensions.

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Consider a $D+1$ dimensional supergravity with a global symmetry G. An element g of the symmetry group acts on a generic field ψ as $\psi \to g[\psi]$. Consider now a dimensional reduction of the theory to D dimensions on a circle of radius R with a periodic coordinate $y \sim y+1$. In the twisted reduction, the fields are not independent of the internal coordinate but are chosen to have a specific dependence on the circle coordinate y through the ansatz

$$\psi(x^\mu, y) = g(y)\,[\psi(x^\mu)] \tag{1.1}$$

for some y-dependent group element $g(y)$ [6]. An important restriction on $g(y)$ is that the reduced theory in D dimensions should be independent of y. This is achieved by choosing

$$g(y) = \exp(My) \tag{1.2}$$

for some Lie-algebra element M. The map $g(y)$ is not periodic around the circle, but has a *monodromy*

$$\mathcal{M}(g) = \exp M\,. \tag{1.3}$$

Many supergravity theories in $D+1 = 2n$ dimensions have a set of n form field strengths H_n^i where $i = 1, \ldots, r$ labels the potentials, which typically satisfy a generalised self-duality equation of the form

$$H_n^i = Q^i{}_j(\phi) * H_n^j \tag{1.4}$$

where $Q^i{}_j$ is a matrix depending on the scalar fields ϕ and $*$ is the Hodge dual in $D+1$ dimensions [21]. For any n, consistency requires that $(Q^i{}_j(\phi)*)^2 = 1$, so that if $(*)^2 = -1$, as in lorentzian space of dimension $4m$, then $Q^2 = -\mathbb{1}$ and Q is a complex structure, while if $(*)^2 = 1$, as in lorentzian space of dimension $4m+2$, then $Q^2 = \mathbb{1}$ and Q is a product structure. In the theories we will consider, the H_n^i transform in an r-dimensional representation of a rigid duality group G. In $d=4$, $N=8$ supergravity, there are $r=56$ 2-form field strengths transforming as a **56** of the duality group $G = E_7$ [22, 23]. These split into 28 field strengths $F = dA$ and 28 dual field strengths $\tilde{F} = \hat{*}F + \ldots$, with Q a complex structure on $\mathbb{R}^{56}$. In $d=6$, $N=8$ supergravity, there are 5 3-form field strengths which split into 5 self-dual ones and 5 anti-self dual ones, and these 10 transform as a **10** of $G = \mathrm{SO}(5,5)$ [24]. The 10 3-form field strengths $\hat{H}_n^i$ with $i = 1, \ldots, 10$, satisfy (anti) self-duality constraints of the form (1.4) with Q related to the $\mathrm{SO}(5,5)$-invariant metric. In $d=8$ maximal supergravity, there is a 3-form potential, and its field strength and its dual combine into an $\mathrm{SL}(2,\mathbb{R})$ doublet, satisfying a constraint of the form (1.4) with $Q = i\sigma_2$.

Our main interest here is in reductions in which the monodromy $\mathcal{M} \in G$ is a symmetry of the equations of motion but not the action, acting on the field strengths $\hat{H}_n^i$ via transformations involving Hodge or electromagnetic dualities, so that they cannot be realised locally on the fundamental $n-1$ form potentials. We find that (in the case in which M is invertible) the field strengths $\hat{H}_n^i$ satisfying the constraint (1.4) give rise to r $n-1$ form potentials A_{n-1}^i in $2n-1$ dimensions satisfying massive self-duality constraints of the form

$$DA_{n-1} = \tilde{M} * A_{n-1}\,, \tag{1.5}$$

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where D is a gauge-covariant exterior derivative, $*$ is now the Hodge dual in D dimensions and the matrix $\tilde{M} \propto QM$. Such odd-dimensional self-duality conditions were first considered in [26] and often occur in odd-dimensional gauged supergravity theories, and follow from a Chern-Simons action with mass term of the form

$$L = P_{ij} A^i \wedge DA^j + \hat{M}_{ij} A^i \wedge *A^j \,, \tag{1.6}$$

where $\hat{M} = P\tilde{M}$ and P_{ij} is a suitably chosen constant matrix. In the general case in which M is not invertible, some of the gauge fields remain massless.

In dimensionally reducing a theory with a twist that is a symmetry of the equations of motion and not of the action, one needs to reduce the equations of motion, not the action. However, for the cases of interest here there is a doubled formalism [21] in which dual potentials $\tilde{A}_{n-1}$ are introduced for each $n-1$ form potential A_{n-1}, in which the duality symmetry becomes a symmetry of the action $S[A, \tilde{A}]$, which is supplemented by a duality-invariant constraint that could be used to eliminate $\tilde{A}$ in terms of A. This doubled action and constraint can then be dimensionally reduced in the standard way with a twist by the duality symmetry. This greatly simplifies the calculations.

We apply these results to the reduction of supergravity theories in $4, 6, 8$ dimensions, giving rise to supergravity theories in $3, 5, 7$ dimensions with massive self-dual forms. This constructs new supergravity theories in these dimensions and gives a higher-dimensional origin for theories in $3, 5, 7$ dimensions with Chern-Simons actions. In particular, for $D = 3$, A is a vector field and this gives a higher dimensional origin for 3-dimensional gauged supergravity theories, of the type discussed in [27] with Chern-Simons actions for some of the gauge fields.

The plan of the paper is as follows. In section 2 we review the Scherk-Schwarz mechanism, giving the results for the twisted reduction of gravity coupled to scalars and gauge potentials, which are used in later sections. We give a detailed analysis of the general case in which the mass matrix is not invertible. In section 3 we review the doubled formalism of [21]. In section 4 we perform a twisted dimensional reduction in the doubled formalism, and hence obtain the lagrangian for dimensional reductions with S-duality twists. Finally, in section 5, we apply our results to the reduction of supergravity theories in $4, 6, 8$ dimensions.

2. Scherk Schwarz reduction

We will consider here Scherk-Schwarz dimensional reduction on a circle from $D+1$ to D dimensions, with a twist by an element of a global symmetry G. The ansatz for dimensional reduction of a generic field is (1.1) with y-dependence given by (1.2) with monodromy $\mathcal{M}$ given by (1.3) in terms of the mass-matrix M. The mass matrix M introduces mass parameters into the theory, and fields in non-trivial representations of the group G typically become massive with masses given in terms of M, or are "eaten" by gauge fields that become massive in a generalised Higgs mechanism. In particular, the scalar fields will obtain a scalar potential given in terms of M. However, different mass-matrices can give equivalent theories, and an important question is how to classify the inequivalent theories.

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In [14] it was shown that the theories are determined by the monodromy $\mathcal{M}$, not the mass matrix M. Two reductions with different mass matrices M, M' but the same monodromy $\mathcal{M} = e^M = e^{M'}$ give the same reduced theory, provided the full spectrum of massive states is kept, and no truncation is made. In [6], it was shown that theories with monodromies in the same G conjugacy class are equivalent, so that the theories are classified by the G conjugacy classes. In quantum string theory, a global group of the classical theory typically becomes a discrete gauge symmetry $G(\mathbb{Z})$ [28] and for such theories the monodromy must be in $G(\mathbb{Z})$, giving quantization conditions on the mass parameters, and the distinct theories are determined by the monodromy $\mathcal{M} \in G(\mathbb{Z})$ up to $G(\mathbb{Z})$ conjugation. The mass matrix M generates a one dimensional subgroup L of G, which becomes a gauge symmetry of the reduced theory, so that such a reduction of a supergravity gives a gauged supergravity [7]–[9] and [14].

Our main interest will be in the reduction of supergravity and superstring theories. Extended supergravity theories typically have a global symmetry G and the scalars take values in the coset space G/H where G is a non-compact group and H is the maximal compact subgroup of G. The scalar sector of the theory is then invariant under the group G and this symmetry typically extends to the full theory for supergravities in odd dimensions. In some even dimensional theories, the symmetry G extends to a symmetry of the equations of motion only, acting through duality transformations exchanging field equations with Bianchi identitites.

The theory can be formulated with a local H symmetry as well as a global G symmetry. The scalars in the coset space G/H can be represented by a vielbein $\mathcal{V}(x) \in G$ which transforms under global G and local H transformations as

$$\begin{aligned} &\mathcal{V} \to h(x)\mathcal{V}\,, \qquad h(x) \in H \\ &\mathcal{V} \to \mathcal{V}\, g\,, \qquad g \in G\,. \end{aligned} \tag{2.1}$$

The lagrangian is

$$L = -\frac{1}{2}\,\mathrm{tr}\left[d\mathcal{V}\mathcal{V}^{-1} \wedge *d\mathcal{V}\mathcal{V}^{-1}\right]. \tag{2.2}$$

In this formulation there are an extra $\dim(H)$ non-physical scalars which can be gauged away using the local H symmetry. Here $\mathcal{V}, g, h$ can be taken to be matrices in some representation of G. We will present our results for real representations of G such that the representatives of H are orthogonal matrices $h^T h = \mathbb{1}$ so that δ_{ab} is an invariant, but the generalisation to other representations is straightforward.

An alternative formulation that does not involve extra scalars is to use a metric $\mathcal{K}$ on G/H instead of a vielbein, transforming as (for a real representation of G)

$$\mathcal{K} \to g^T \mathcal{K} g\,. \tag{2.3}$$

Such a metric can be constructed from the vielbein as $\mathcal{K}_{ij} = \delta_{ab}\mathcal{V}^a{}_i\mathcal{V}^b{}_j$, where i and a are the curved and flat indices respectively. $\mathcal{K}$ is invariant under local H transformations as $h^T h = \mathbb{1}$. This means that the non-physical scalars drop out in this formulation, without any need for gauge-fixing. (For complex representations with $h^\dagger h = \mathbb{1}$, we would use the hermitean metric $\mathcal{K} = \mathcal{V}^\dagger \mathcal{V}$ transforming as $\mathcal{K} \to g^\dagger \mathcal{K} g$.) The lagrangian can be written in terms of $\mathcal{K}$ as

$$L = \frac{1}{4}\,\mathrm{tr}[d\mathcal{K}^{-1} \wedge * d\mathcal{K}]\,. \tag{2.4}$$

An example which will play a central role in what follows is a theory of gravity coupled to scalars in the coset G/H and a set of r $n-1$ form gauge potentials A^i_{n-1} with n-form field strengths $H^i_n = dA^i_{n-1}$ (where $i = 1, \ldots, r$) transforming in a real r-dimensional representation of the symmetry group G. We take $\mathcal{V}$ to be an $r \times r$ matrix acting in the r-dimensional representation of G and consider the theory in $D+1$ dimensions and work with the metric $\mathcal{K}_{ij}$. The lagrangian is

$$\mathcal{L} = R * 1 + \frac{1}{4}\,\mathrm{tr}(d\mathcal{K} \wedge * d\mathcal{K}^{-1}) - \frac{1}{2} H_n^T \mathcal{K} \wedge * H_n\,. \tag{2.5}$$

The action is invariant under the rigid G symmetry

$$\delta A \to L^{-1} A, \qquad \delta\mathcal{K} \to L^T \mathcal{K} L\,, \tag{2.6}$$

where $L^i{}_j$ is a G-transformation in the $\mathbf{r}$ representation, and the spacetime metric is invariant. In later sections, we will be particularly interested in the case in which $D+1 = 2n$, but for now we will keep D, n arbitrary.

For example, in the case $G = \mathrm{SL}(2,\mathbb{R})$, $H = \mathrm{SO}(2)$, there are two scalars in the theory, which we will denote ϕ and χ, which parametrise the scalar coset $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$. The matrix $\mathcal{V}$ (in the doublet representation of $\mathrm{SL}(2,\mathbb{R})$) is a general $\mathrm{SL}(2,\mathbb{R})$ matrix, which can be given, in terms of ϕ and χ and a non-physical scalar θ that parameterises the $\mathrm{SO}(2)$ subgroup, by

$$\mathcal{V} = h e^{\phi/2} \begin{pmatrix} e^{-\phi} & 0 \\ -\chi & 1 \end{pmatrix} \tag{2.7}$$

where h is an $\mathrm{SO}(2)$ matrix

$$h = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{2.8}$$

Then

$$\mathcal{K} = e^{\phi} \begin{pmatrix} e^{-2\phi} + \chi^2 & -\chi \\ -\chi & 1 \end{pmatrix} \tag{2.9}$$

and the lagrangian (2.5) can be written as

$$\mathcal{L} = R{*}1 - \frac{1}{2} d\phi \wedge * d\phi - \frac{1}{2} e^{2\phi} d\chi \wedge * d\chi - \frac{1}{2}(e^{-\phi} + e^{\phi}\chi^2) H^1 \wedge * H^1 - \frac{1}{2} e^{\phi} H^2 \wedge * H^2 - \chi e^{\phi} H^1 \wedge * H^2 \tag{2.10}$$

and is independent of θ.

We now reduce the lagrangian (2.5) on a circle with a twist given by a monodromy $\mathcal{M} = e^M \in G$ with the ansatz (1.1). For the remainder of this section, we distinguish the $D+1$-dimensional fields from D-dimensional ones by a hat. The metric is invariant under the global symmetry group so we use the standard Kaluza-Klein ansatz

$$d\hat{s}^2 = e^{2\alpha\varphi} ds^2 + e^{2\beta\varphi}(dy + \mathcal{A})^2 \tag{2.11}$$

so that the Einstein-Hilbert term in (2.5) reduces to

$$\mathcal{L}_g = R * 1 - \frac{1}{2} d\varphi \wedge * d\varphi - e^{-2(D-1)\alpha\varphi} \frac{1}{2} \mathcal{F} \wedge * \mathcal{F} \,. \tag{2.12}$$

Here φ is the scalar field coming from the reduction of the metric, $\mathcal{F} = d\mathcal{A}$ and $\mathcal{A}$ is the graviphoton. The constants α and β depend on D and are:

$$\alpha^2 = \frac{1}{2(D-1)(D-2)} \,, \qquad \beta = -(D-2)\alpha \,. \tag{2.13}$$

From (1.1) and (2.6) the ansatz for the scalar fields and the 3-form fields is

$$\hat{\mathcal{K}}(x,y) = \lambda^T(y) \mathcal{K}(x) \lambda(y) \tag{2.14}$$

$$\hat{A}_{n-1}(x,y) = \lambda^{-1}(y) [A_{n-1}(x) + A_{n-2}(x) \wedge dy] \,, \tag{2.15}$$

where $\lambda(y) = e^{My}$.

By using the ansatz (2.14) one finds that the reduction of the scalar kinetic term $\frac{1}{4} \operatorname{tr}(d\hat{\mathcal{K}} \wedge * d\hat{\mathcal{K}}^{-1})$ from $D+1$ dimensions to D dimensions gives a scalar kinetic term plus a scalar potential [1]:

$$\mathcal{L}_s = \frac{1}{4} \operatorname{tr}(\mathcal{D}\mathcal{K} \wedge * \mathcal{D}\mathcal{K}^{-1}) + V(\phi) \,, \tag{2.16}$$

where

$$\begin{aligned} \mathcal{D}\mathcal{K} &= d\mathcal{K} - (M^T \mathcal{K} + \mathcal{K} M) \wedge \mathcal{A} \\ \mathcal{D}\mathcal{K}^{-1} &= d\mathcal{K}^{-1} + (M \mathcal{K}^{-1} + \mathcal{K}^{-1} M^T) \wedge \mathcal{A} \end{aligned} \tag{2.17}$$

and the scalar potential $V(\phi)$ is

$$V(\phi) = -\frac{1}{2} e^{2(D-1)\alpha\varphi} \operatorname{tr}(M^2 + M \mathcal{K}^{-1} M^T \mathcal{K}) * 1 \,. \tag{2.18}$$

The ansatz (2.15) implies

$$\hat{H}_n(x,y) = e^{-My} H_n(x) + e^{-My} H_{n-1}(x) \wedge (dy + \mathcal{A}) \tag{2.19}$$

for the n-form field strengths $\hat{H}_n = d\hat{A}_{n-1}$. Here the D-dimensional field strengths are

$$H_{n-1}(x) = dA_{n-2} - (-1)^{n-1} M A_{n-1} \,, \qquad H_n(x) = dA_{n-1} - H_{n-1} \wedge \mathcal{A} \,. \tag{2.20}$$

Reduction of the kinetic term gives

$$\hat{H}_n^T \hat{\mathcal{K}} \wedge \hat{*} \hat{H}_n \to [e^{-2(n-1)\alpha\varphi} H_n^T \mathcal{K} \wedge * H_n + e^{2(D-n)\alpha\varphi} H_{n-1}^T \mathcal{K} \wedge * H_{n-1}] \wedge dy \,. \tag{2.21}$$

Collecting the results we can now write down the D-dimensional lagrangian as:

$$\mathcal{L}_D = \mathcal{L}_g + \mathcal{L}_b + \mathcal{L}_s \,, \tag{2.22}$$

where

$$\begin{aligned} \mathcal{L}_g &= R * 1 - \frac{1}{2} d\varphi \wedge * d\varphi - \frac{1}{2} e^{-2(D-1)\alpha\varphi} \mathcal{F}_2 \wedge * \mathcal{F}_2 \\ \mathcal{L}_s &= \frac{1}{4} \operatorname{tr}(\mathcal{D}\mathcal{K} \wedge * \mathcal{D}\mathcal{K}^{-1}) - \frac{1}{2} e^{2(D-1)\alpha\varphi} \operatorname{tr}(M^2 + M\mathcal{K}^{-1} M^T \mathcal{K}) * 1 \end{aligned} \tag{2.23}$$

and

$$\mathcal{L}_b = -\frac{1}{2}e^{-2(n-1)\alpha\varphi}H_n^T\mathcal{K}\wedge *H_n - \frac{1}{2}e^{2(D-n)\alpha\varphi}H_{n-1}^T\mathcal{K}\wedge *H_{n-1}\,. \tag{2.24}$$

The field strengths (2.20) are invariant under the following gauge transformations:

$$\delta A_{n-1} = d\Lambda\,, \qquad \delta A_{n-2} = (-1)^{n-1}M\Lambda\,. \tag{2.25}$$

If M is invertible, these can be used to gauge A_{n-2} to zero by performing the gauge transformation:

$$A_{n-1} \to A_{n-1} + (-1)^{n-1}M^{-1}dA_{n-2}\,. \tag{2.26}$$

In this gauge the D-dimensional field strengths become

$$H_n = DA_{n-1} = dA_{n-1} - (-1)^n MA_{n-1}\wedge\mathcal{A} \tag{2.27}$$

$$H_{n-1} = (-1)^n MA_{n-1}\,. \tag{2.28}$$

Then A_{n-2} disappears from the theory, and the term $H_{n-1}\wedge *H_{n-1}$ is a mass term for A_{n-1}. The degrees of freedom represented by the r fields A_{n-2} have been absorbed by the r $(n-1)$-form fields A_{n-1} which have become massive. Now $H_n = DA_{n-1}$ is a gauge covariant derivative where the gauge group is the subgroup of G generated by M and the corresponding gauge field is the graviphoton $\mathcal{A}$.

Now we will analyze the case M is not invertible. It is useful to work with flat indices $H^a = \mathcal{V}^a_iH^i$, $A^a = \mathcal{V}^a_iA^i$. Then $H^a = \mathcal{D}A^a = dA^a + \omega^a{}_bA^b$ where ω is the connection 1-form $\omega^a_{\ b} = \mathcal{V}^a_i(d\mathcal{V}^{-1})^i{}_b$. The groups G arising in the supergravity theories of interest here all have a G-invariant matrix Ω which is symmetric if n is odd and anti-symmetric if n is even

$$\Omega^{ab} = (-1)^{n-1}\Omega^{ba}\,. \tag{2.29}$$

Using this, we introduce $\bar{H}_a = (\Omega^{-1})_{ab}H^b$ and $M^{ab} = M^a{}_c\Omega^{cb}$. Now one has

$$H^a_{n-1} = \mathcal{D}A^a_{n-2} - (-1)^{n-1}M^{ab}\bar{A}_{(n-1)b}\,, \qquad \bar{H}_{(n)a} = \mathcal{D}\bar{A}_{(n-1)a} - \bar{H}_{(n-1)a}\wedge\mathcal{A} = \tilde{\mathcal{D}}\bar{A}_{(n-1)a}\,, \tag{2.30}$$

where $\tilde{\mathcal{D}}$ is the covariant derivative with connections $\omega^a{}_b$ and $\mathcal{A}$.

Note that $\mathcal{M} = e^M$ and $\mathcal{M}^T\Omega^{-1}\mathcal{M} = \Omega^{-1}$ since $\mathcal{M}\in G$ and Ω is G-invariant. (For complex representations, the condition is $\mathcal{M}^\dagger\Omega^{-1}\mathcal{M} = \Omega^{-1}$.) As a result the mass matrix $M^a{}_b$ satisfies:

$$M^T\Omega^{-1} + \Omega^{-1}M = 0\,. \tag{2.31}$$

From (2.29) and (2.31) it follows that M^{ab} is a symmetric matrix if n is even and antisymmetric if n is odd:

$$M^{ab} = (-1)^n M^{ba}\,. \tag{2.32}$$

Let the dimension of $ker(M)$ be l. Now the matrix M^{ab} can be brought into the canonical form

$$M^{ab} = \begin{pmatrix} 0 & 0 \\ 0 & m^{\alpha'\beta'}\end{pmatrix}, \tag{2.33}$$

where $m^{\alpha'\beta'}$ is an invertible $(r-l)\times(r-l)$ matrix which is diagonal if n is even and skew-diagonal if n is odd. Here we have split the indices $a\to(\alpha,\alpha')$ where α runs from 1 to l and α' runs from $l+1$ to r. Similarly the gauge fields A can be written in the block form

$$A = \begin{pmatrix} A^{\alpha} \\ A^{\alpha'} \end{pmatrix} . \tag{2.34}$$

Performing the gauge transformation

$$\bar{A}_{(n-1)\alpha'} \to \bar{A}_{(n-1)\alpha'} + (-1)^{n-1}(m^{-1})_{\alpha'\beta'} \mathcal{D} A^{\beta'}_{n-2} \tag{2.35}$$

one sees that the $r-l$ fields $\bar{A}_{(n-1)\alpha'}$ become massive, having eaten the $r-l$ fields $A^{\alpha'}_{n-2}$, while A^{α}_{n-2} and $\bar{A}_{(n-1)\alpha}$ both remain in the theory as massless gauge fields, with l of each. The field strengths for the $(n-2)$-form fields in (2.30) become

$$H^{\alpha'}_{n-1} = (-1)^n m^{\alpha'\beta'} \bar{A}_{(n-1)\beta'} \,, \qquad H^{\alpha}_{n-1} = \mathcal{D} A^{\alpha}_{n-2} \tag{2.36}$$

and hence the term (2.24) can be written as

$$\begin{aligned} \mathcal{L}_b = & -\frac{1}{2} e^{-2(n-1)\alpha\varphi} \delta^{\alpha\beta} \bar{H}_{(n)\alpha} \wedge *\bar{H}_{(n)\beta} - \frac{1}{2} e^{-2(n-1)\alpha\varphi} \delta^{\alpha'\beta'} \bar{H}_{(n)\alpha'} \wedge *\bar{H}_{(n)\beta'} - \\ & - \frac{1}{2} e^{2(D-n)\alpha\varphi} \delta_{\alpha\beta} \mathcal{D} A^{\alpha}_{n-2} \wedge *\mathcal{D} A^{\beta}_{n-2} - \frac{1}{2} e^{2(D-n)\alpha\varphi} (m^T m)^{\alpha'\beta'} \bar{A}_{(n-1)\alpha'} \wedge *\bar{A}_{(n-1)\beta'} \,. \end{aligned} \tag{2.37}$$

We have chosen the normalisation of Ω so that $\Omega^{ac}\Omega^{bd}\delta_{cd} = \delta^{ab}$.

The gauge group, the couplings and the scalar potential of the D-dimensional theory found above are given explicitly in terms of the mass matrix M, and two theories are distinct if the monodromies are in distinct G-conjugacy classes. For the case $G = \mathrm{SL}(2,\mathbb{R})$ there are three conjugacy classes, the hyperbolic, elliptic and parabolic conjugacy classes and so there are three distinct reductions [6]. The hyperbolic, elliptic and parabolic monodromy matrices and mass matrices can be taken to be:

$$\mathcal{M}_h = \begin{pmatrix} e^m & 0 \\ 0 & e^{-m} \end{pmatrix}, \qquad \mathcal{M}_e = \begin{pmatrix} \cos m & \sin m \\ -\sin m & \cos m \end{pmatrix}, \qquad \mathcal{M}_p = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \tag{2.38}$$

$$M_h = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}, \qquad M_e = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}, \qquad M_p = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}. \tag{2.39}$$

The mass matrix generates a one-parameter subgroup of $\mathrm{SL}(2,\mathbb{R})$ and this subgroup will be the gauge group in the lower dimensional theory. Thus compactification with M_e will give a compact gauging SO(2) whereas compactification with M_h and M_p will give rise to SO(1,1)-gauged lower dimensional theories [7]. (Note that in the special case in which $n=3$, there will be extra vector gauge fields in D dimensions from the reduction of the 2-form gauge fields, and strictly speaking the gauge group is $ISO(2)$, $ISO(1,1)$ or the Heisenberg group for the elliptic, hyperbolic and parabolic cases, respectively [14].)

The parabolic mass matrix M_p is not invertible, and has a one-dimensional kernel, i.e. $r=2, l=1$, so that α and α' both take only one value and $A^a = (A^1, A^{1'})$. In this case the matrix $m^{\alpha'\beta'}$ in (2.33) is the 1×1 matrix $(-m)$ and from the gauge transformation (2.35) it can be seen that the $(n-1)$-form $\bar{A}_{n-1\ 1'}$ eats the $(n-2)$-form $A^{1'}_{n-2}$ and becomes massive. The remaining $n-1$ form $\bar{A}_{n-1\ 1}$ and $n-2$ form A^{1}_{n-2} gauge fields remain massless.

3. The doubled formalism

Typically a $D = 2n$ dimensional supergravity theory has a global symmetry group G which can be realised at the level of field equations but not the action, as G acts on n-form field strengths $H = dA$ through electric-magnetic duality transformations. In such cases it is possible to construct a manifestly G-invariant lagrangian that depends on the potentials A and dual potentials $\tilde{A}$. The dual fields are regarded as independent fields, but the field equations are supplemented with a G-covariant constraint relating the n-form field strengths $d\tilde{A}$ to dA, keeping the number of independent degrees of freedom correct. The new lagrangian is equivalent to the original one as the two yield equivalent field equations when the constraint is taken into account.

In this section we will review this formalism, which was introduced in [21] where it was called the 'doubled formalism'. We will first consider the case $G = \mathrm{SL}(2, \mathbb{R})$ and then give the general case in the following subsection.

3.1 $G = \mathrm{SL}(2, \mathbb{R})$ case

Consider the following lagrangian in $2n$ dimensions with n even

$$\mathcal{L} = -\frac{1}{2} d\phi \wedge * d\phi - \frac{1}{2} e^{2\phi} d\chi \wedge * d\chi - \frac{1}{2} e^{-\phi} F_n \wedge * F_n - \frac{1}{2} \chi F_n \wedge F_n \,. \tag{3.1}$$

Here $F_n = dA_{n-1}$ and ϕ and χ are scalar fields. The field equations of this lagrangian have an $\mathrm{SL}(2, \mathbb{R})$ S-duality invariance (for even n) acting on F through electromagenetic duality transformations, as we now discuss.

Defining a new n-form G by

$$G_n = \frac{\delta \mathcal{L}}{\delta F_n} = -e^{-\phi} * F_n - \chi F_n \tag{3.2}$$

the lagrangian (3.1) can be written as

$$\mathcal{L} = \frac{1}{4} \mathrm{tr}(d\mathcal{K} \wedge * d\mathcal{K}^{-1}) + \frac{1}{2} F_n \wedge G_n \,, \tag{3.3}$$

where $\mathcal{K}$ is as in (2.9). The Bianchi identity and the equation of motion for the n-form field strength F_n are

$$\begin{aligned} dF_n &= 0 \\ dG_n = d(-e^{-\phi} * F_n - \chi F_n) &= 0 \end{aligned} \tag{3.4}$$

which can be combined as

$$d\mathcal{H}_n = 0 \,, \tag{3.5}$$

where $\mathcal{H}_n$ is the $\mathrm{SL}(2, \mathbb{R})$ doublet

$$\mathcal{H}_n = \begin{pmatrix} F_n \\ G_n \end{pmatrix} . \tag{3.6}$$

The field equations are manifestly SL$(2,\mathbb{R})$ invariant, but the $F_n \wedge G_n$ term in the lagrangian (3.3) is not invariant. However, an invariant lagrangian can be constructed as in [21] if the field equation $dG_n = 0$ is solved by introducing a dual potential $\tilde{A}_n$ so that $G_n = d\tilde{A}_n$, which can be combined with A_n to form an SL$(2,\mathbb{R})$ doublet, with field strengths H_n^i given by

$$H_n = \begin{pmatrix} dA_n \\ d\tilde{A}_n \end{pmatrix} . \tag{3.7}$$

Then the natural SL$(2,\mathbb{R})$ invariant lagrangian is

$$\mathcal{L}' = \frac{1}{4}\,\mathrm{tr}(d\mathcal{K} \wedge *d\mathcal{K}^{-1}) - \frac{1}{4} H_n^i \mathcal{K}_{ij} \wedge * H_n^j \tag{3.8}$$

which is of the form considered in the previous section.

For this action, both A_{n-1} and $\tilde{A}_{n-1}$ are independent fields, so that the number of $n-1$ form degrees of freedom has been doubled. To halve them again, for even n this action can be supplemented by the SL$(2,\mathbb{R})$ covariant constraint [21]

$$H_n^i = J^i{}_j * H_n^j \,, \tag{3.9}$$

where J is the SL$(2,\mathbb{R})$ matrix

$$J^i{}_j = \Omega^{ik}\mathcal{K}_{kj} \,. \tag{3.10}$$

Here Ω is the SL$(2,\mathbb{R})$ invariant matrix

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{3.11}$$

Note that the matrix J in (3.10) satisfies $J^2 = -\mathbb{1}$, so that this constraint is consistent in $2n$ dimensions with even n in which $(*)^2 = -1$. The general case including odd n will be discussed in the next subsection.

The field equations derived from (3.8) are supplemented by the extra condition (3.9). Then the field equations derived from (3.8), together with the constraint (3.9) which can be used to rewrite all terms involving $\tilde{A}$ in terms of A, gives precisely the field equations derived from the original action, so that the original lagrangian (3.1) and the SL$(2,\mathbb{R})$ invariant lagrangian (3.8) and constraint (3.9) are equivalent. Note that the conventional normalisation of the gauge field kinetic term in (3.8) has a factor of 1/4 in the doubled formalism whereas in equation (2.5) it had a factor of 1/2. Similar factors of 1/4 will occur in the normalisations of kinetic terms in subsequent lagrangians in the doubled formalism.

3.2 The general formalism

The doubled formalism of the last section can be generalised [21]. Consider the following lagrangian in $2n$ dimensions

$$\mathcal{L} = -\frac{1}{2} R_{IJ} F_n^I \wedge * F_n^J - \frac{1}{2} S_{IJ} F_n^I \wedge F_n^J + \mathrm{L}(\Phi) \,, \tag{3.12}$$

where $F_n^I = dA_{n-1}^I$ with $I = 1, \ldots, k$ are k field strengths and Φ denotes all the remaining fields, including scalar fields. The matrices R_{IJ}, S_{IJ} are functions of the scalar fields and they satisfy $R_{IJ} = R_{JI}$ and $S_{IJ} = (-1)^{n-1} S_{JI}$. It is useful to define G_n^I as

$$G_n^I = \frac{\delta \mathcal{L}}{\delta F_n^I}. \tag{3.13}$$

so that the lagrangian can be written as

$$\mathcal{L} = \frac{1}{2} F_n^I \wedge G_n^I + \mathrm{L}(\Phi)\,. \tag{3.14}$$

The field equations and Bianchi identities can be combined as

$$d\mathcal{H}_n = 0\,, \tag{3.15}$$

where $\mathcal{H}_n$ is

$$\mathcal{H}_n = \begin{pmatrix} F_n^I \\ G_n^I \end{pmatrix} \tag{3.16}$$

with $r = 2k$ components.

Such systems arise in supergravity theories, and typically the field equations and Bianchi identities have a global symmetry G under which $\mathcal{H}_n$ transforms as a $2k$-dimensional representation of G, and $\mathrm{L}(\Phi)$ is G-invariant. The group G has a constant invariant matrix $\Omega^{ij} = \Omega^{ab}(\mathcal{V}^{-1})^i{}_a(\mathcal{V}^{-1})^j{}_b$, where $i, j = 1, \ldots, 2k$ are indices for the **2k** representation of G, satisfying $\Omega^{ij} = (-1)^{n-1}\Omega^{ji}$.

As before we introduce potential fields $\tilde{A}_{n-1}^I$ with $G_n^I = d\tilde{A}_{n-1}^I$ to form

$$H_n = \begin{pmatrix} dA_n^I \\ d\tilde{A}_n^I \end{pmatrix} \tag{3.17}$$

transforming in the **2k** representation of G. Then the system can be described by the G-invariant lagrangian

$$\mathcal{L}' = -\frac{1}{4} H_n^T \mathcal{K} \wedge * H_n + \mathrm{L}(\Phi), \tag{3.18}$$

together with a constraint

$$H_n = Q * H_n\,, \tag{3.19}$$

where $Q^i{}_j$ is a $2k \times 2k$ matrix given in terms of the scalar fields by

$$Q^i{}_j = \Omega^{ik}\mathcal{K}_{kj}\,. \tag{3.20}$$

Here $\mathcal{K}_{ij}$ is given in terms of R_{IJ}, S_{IJ} by

$$\mathcal{K} = \begin{pmatrix} R + SR^{-1}S^T & -SR^{-1} \\ -R^{-1}S^T & R^{-1} \end{pmatrix}. \tag{3.21}$$

In the supergravity applications we will be considering, the scalars take values in a coset G/H and $\mathcal{K}_{ij}$ is the symmetric matrix representing the scalar fields, as described in section 2. Note that

$$Q^2 = (\Omega\mathcal{K})^2 = (-1)^{n-1}\mathbb{1}. \tag{3.22}$$

so that the constraint (3.19) is consistent as for $2n$-dimensional lorentzian space-time $**H_n = (-1)^{n-1} H_n$. It was shown in [21] that the field equations from (3.18) are equivalent to those from (3.12) together with the constraint (3.19).

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4. Reduction with duality twist

The theory with lagrangian (3.12) has a global symmetry G of the equations of motion which acts via duality transformations. In this section we will dimensionally reduce on a circle from $D+1=2n$ to D dimensions with a twist that has monodromy $\mathcal{M}$ in G. For some choices of monodromy $\mathcal{M}$ in G, this is in fact a symmetry of the action and this is a standard Scherk-Schwarz reduction, as in section 2. If it is only a symmetry of the equations of motion, then we use the doubled formalism of section 3 with lagrangian (3.18) supplemented by the constraint (3.19). The lagrangian (3.18) is of the same form as (2.5), so the Scherk-Schwarz reduction of the action proceeds as in section 2. This is supplemented by the constraints arising from the dimensional reduction of (3.19). The field equations in $2n-1$ dimensions are then those from the reduced action together with the reduced constraints, and we go on to seek an action in $2n-1$ dimensions that gives both the constraints and the reduced field equations.

4.1 Dimensional reduction in the doubled formalism

The lagrangian (3.18) in the doubled formalism is of the same form as (2.5), but with an extra factor of 1/2 in the normalisation of the gauge field kinetic term. The Scherk-Schwarz reduction of the lagrangian (2.5) was already discussed in section 2, where we showed that it yields the lagrangian (2.22) in D dimensions. It follows that the reduction of (3.18) should give (2.22) but now with (2.24) divided by two to give:

$$\mathcal{L}_b = -\frac{1}{4}e^{-2(n-1)\alpha\varphi} H_n^T \mathcal{K}\wedge * H_n - \frac{1}{4}e^{2(D-n)\alpha\varphi} H_{n-1}^T \mathcal{K}\wedge * H_{n-1}\,. \tag{4.1}$$

Just as the lagrangian (3.18) should be supplemented by the $D+1$ dimensional constraint (3.19) in order to give the correct $D+1$ dimensional field equations, the D dimensional lagrangian (2.22) with (2.23), (4.1) should be supplemented by the constraint which is obtained by the dimensional reduction of (3.19). In this section we will describe the reduction of the $D+1$-dimensional constraint (3.19). Note that it is G-covariant, so the y dependence of the fields in the ansatz (1.1) cancels out in the reduction.

Using the ansatz (2.14), (2.15) the $D+1$ dimensional constraint (3.19) reduces to the D-dimensional constraint:

$$H_n = e^{\gamma} Q * H_{n-1}\,, \tag{4.2}$$

where Q is as in (3.20), $\mathcal{K}$ is given by (3.21) and we have defined $\gamma \equiv 2(D-n)\alpha\varphi$. As a result, the n-form field strengths are dual to the $n-1$-form field strengths. The constraint (4.2) can be rewritten using flat indices as

$$\bar{H}_{(n)a} = e^{\gamma}\delta_{ab} * (\mathcal{D}A^b_{(n-2)} + (-1)^n M^{bc}\bar{A}_{(n-1)c})\,. \tag{4.3}$$

For an untwisted reduction (i.e. one with $M=0$, so that it is a standard reduction) this constraint can be used to eliminate the $2k$ potentials A_{n-1} so that the theory can be written in terms of the $2k$ potentials A_{n-2} (or alternatively the potentials A_{n-2} can be eliminated and the theory written in terms of the A_{n-1}, or more generally in terms of s potentials A_{n-2} and $2k-s$ potentials A_{n-1}). In the twisted case with invertible M, one can go to the gauge in which the fields A^i_{n-2} are set zero, as was discussed in section 2. In this gauge the field strengths H_n and H_{n-1} are given in (2.27) and (2.28) so that the duality condition (4.2) is:

$$DA_{n-1} = (-1)^n e^\gamma \tilde{M} * A_{n-1}\,, \tag{4.4}$$

where $\tilde{M} = QM$. This is a massive self-duality condition for the $2k$ potentials A_{n-1}. Such self-duality conditions in odd dimensions were introduced in [26]. The self-duality constraint (4.4) implies the massive field equation (suppressing non-linear terms)

$$*D * DA_{n-1} = e^{2\gamma}\tilde{M}^2 A_{n-1} + \cdots \tag{4.5}$$

with mass matrix proportional to $\tilde{M}^2$. However, the constraint (4.4) halves the number of degrees of freedom of a massive $n-1$ form field.

It is instructive to check the number of physical degrees of freedom. In d dimensions a massless p form gauge field A_p has c_p^{d-2} degrees of freedom, where c_p^s is the binomial coefficient

$$c_p^s = \frac{(s)!}{p!(s-p)!}$$

while a massive p form gauge field has c_p^{d-1} degrees of freedom. The k gauge fields A^I_{n-1} in $2n$ dimensions have kc^{2n-2}_{n-1} degrees of freedom, which can be represented by the $2k$ gauge fields A^i_{n-1} (with $2kc^{2n-2}_{n-1}$ degrees of freedom) together with k constraints (3.19) that halve the number of degrees of freedom again. In an untwisted reduction, each massless $n-1$ form gauge field in $2n$ dimensions gives rise to a massless $n-1$ form gauge field and a massless $n-2$ form gauge field, and the number of degrees of freedom is correct as

$$c^{2n-2}_{n-1} = c^{2n-3}_{n-1} + c^{2n-3}_{n-2}$$

However, the number of degrees of freedom of a massive p-form in $d-1$ dimensions is c_p^{d-2}, which is the same as the number of degrees of freedom of a massless p-form in d dimensions, and in the twisted reduction with invertible M, all the $n-1$ forms in $2n$ dimensions give rise to massive $n-1$ forms in $2n-1$ dimensions. We have $2k$ massive gauge fields A^i_{n-1} in $2n-1$ dimensions which have $2kc^{2n-2}_{n-1}$ degrees of freedom, but the self-duality constraints (4.4) remove half of the degrees of freedom, leaving kc^{2n-2}_{n-1} degrees of freedom, as required.

When M is not invertible, the field strengths are given by (2.36). Then the constraint (4.3) takes the form (dropping the coupling to the graviphoton again)

$$\mathcal{D}\bar{A}_{(n-1)\alpha'} = (-1)^n e^\gamma \delta_{\alpha'\beta'} * m^{\beta'\gamma'}\bar{A}_{(n-1)\gamma'} \tag{4.6}$$

$$\mathcal{D}\bar{A}_{(n-1)\alpha} = e^\gamma \delta_{\alpha\beta} * \mathcal{D}A^\beta_{n-2}\,. \tag{4.7}$$

Before imposing the constraint, the $r-l$ fields $\bar{A}_{(n-1)\alpha'}$ are massive, having eaten the $r-l$ fields $A^{\alpha'}_{n-2}$, while A^α_{n-2} and $\bar{A}_{(n-1)\alpha}$ both remain in the theory as massless gauge fields, with l of each, as was seen in section 2. So before imposing the constraint the total number of degrees of freedom is

$$(r-l)c^{2n-2}_{n-1} + lc^{2n-3}_{n-2} + lc^{2n-3}_{n-1} = rc^{2n-2}_{n-1} = 2kc^{2n-2}_{n-1}.$$

Imposing the constraint imposes self-duality on the massive fields $\bar{A}_{(n-1)\alpha'}$, halving the number of degrees of freedom, and relates $\bar{A}_{(n-1)\alpha}$ to A^α_{n-2}, so that half of them can be eliminated (e.g. A^α_{n-2} can be eliminated leaving $\bar{A}_{(n-1)\alpha}$, or $\bar{A}_{(n-1)\alpha}$ can be eliminated leaving A^α_{n-2}). Thus one is left with kc^{2n-2}_{n-1} degrees of freedom, as required.

The field equations from the D-dimensional lagrangian (2.22) with (2.23), (4.1) are supplemented by the D dimensional constraint (4.2). This implies that the field strengths H_n and H_{n-1} in (4.1) are not independent but are related via the duality condition (4.2). Note that if this constraint were applied to the action, it would make the gauge field kinetic term (4.1) vanish. This was to be expected as the twisted self-duality condition (3.19), from which the duality condition (4.2) is obtained, implies the vanishing of the gauge kinetic term in the $D+1$-dimensional doubled lagrangian (3.18). Thus it is important that one first varies the action and then imposes the constraint (4.2).

It is straightforward to verify that the field equations derived from $\mathcal{L}_b$ for the potentials A_{n-1} are consistent with the D dimensional constraint (4.2). After some computation one finds that the condition for consistency is that the mass matrix M should satisfy the equation (2.31).

4.2 Lagrangian for reduced theory

The odd dimensional massive self-duality condition (4.4) can be obtained from a Chern-Simons action of the form (1.6), as we now show. In the case in which M is invertible, the D-dimensional constraint (4.4) follows from the following lagrangian:

$$\mathcal{L}'_b = \frac{1}{2}P_{ij}[(-1)^{n-1}A^i_{n-1}\wedge DA^j_{n-1} + e^\gamma \tilde{M}^j{}_k A^i_{n-1}\wedge * A^k_{n-1}]\,, \tag{4.8}$$

where P_{ij} is any invertible matrix satisfying $P^T=(-1)^nP$. This generalises the lagrangian of [26]. We now show that for the special choice $P_{ij}=(\Omega^{-1}M)_{ij}$, varying this action with respect to the scalars, metric and other fields also give the right equations of motion. Note that it follows from (2.31) that $P=\Omega^{-1}M$ is a symmetric matrix if n is even and is antisymmetric if n is odd. Similarly, $P\tilde{M}$ is a symmetric matrix.

Now consider the lagrangian

$$\mathcal{L}'_D = \mathcal{L}_g + \mathcal{L}_s + \mathcal{L}'_b\,, \tag{4.9}$$

where $\mathcal{L}_g$ and $\mathcal{L}_s$ are as in (2.23). We will show that $\mathcal{L}'_D$ is equivalent to the lagrangian $\mathcal{L}_D$ (2.22) with $\mathcal{L}_b$ given by (4.1) in the sense that they yield the same field equations for all fields when the field equations of $\mathcal{L}_D$ are supplemented by the D-dimensional constraint; the analysis is similar to that in [21].

The field equations for the potential fields A_{n-1} have already been discussed. Now we check the field equations for the scalar fields. Let κ represent any of the scalar fields in the theory except for the Kaluza-Klein field φ. Then

$$\begin{aligned}\delta_\kappa\mathcal{L}_b &= -\frac{1}{4}e^{-\gamma}H^T_n\frac{\delta\mathcal{K}}{\delta\kappa}\wedge *H_n - \frac{1}{4}e^{\gamma}H^T_{n-1}\frac{\delta\mathcal{K}}{\delta\kappa}\wedge *H_{n-1}\\ &= -\frac{e^\gamma}{4}A^T_{n-1}\tilde{M}^T\frac{\delta\mathcal{K}}{\delta\kappa}\tilde{M}\wedge *A_{n-1} - \frac{e^\gamma}{4}A^T_{n-1}M^T\frac{\delta\mathcal{K}}{\delta\kappa}M\wedge *A_{n-1}\\ &= -\frac{e^\gamma}{2}A^T_{n-1}M^T\frac{\delta\mathcal{K}}{\delta\kappa}M\wedge *A_{n-1} = \delta_\kappa\mathcal{L}'_b\,. \end{aligned}\tag{4.10}$$

In the second line we have imposed the constraint (4.4). In the third line we used the symmetry properties of the matrices Ω and $\mathcal{K}$, the fact that Ω is G-invariant and also that $\frac{\delta\mathcal{K}}{\delta\kappa}\,\Omega\mathcal{K} = -\mathcal{K}\Omega\,\frac{\delta\mathcal{K}}{\delta\kappa}$. The last equality in (4.10) holds because $P\tilde{M} = -M^T\Omega^{-1}\Omega\mathcal{K}M = -M^T\mathcal{K}M$. This establishes that the two lagrangians $\mathcal{L}$ and $\mathcal{L}'$ have the same field equations for the scalar fields.

In order to check the equivalence of the field equations for the metric, it is useful to note the following relation:

$$\frac{\delta}{\delta g^{\alpha\beta}}\left(H_n^T\mathcal{K}\wedge *H_n\right)=\mathcal{K}_{ij}\sqrt{-g}\left(-nH_n^{(i)\alpha\mu_1\cdots\mu_{n-1}}\;H^{(j)\beta}_{n\mu_1\cdots\mu_{n-1}}+\frac{1}{2}g^{\alpha\beta}H_n^{(i)\mu_1\cdots\mu_n}H^{(j)}_{n\;\mu_1\cdots\mu_n}\right). \tag{4.11}$$

Two such terms come from the variation of $\mathcal{L}_b$ in (4.1). Imposing the constraint (4.2) on these terms and then using the properties of the matrices $\mathcal{K}$, M and Ω as before one can show that

$$\begin{aligned}\frac{\delta\mathcal{L}_b}{\delta g^{\alpha\beta}} &= \frac{e^\gamma}{2(n-1)!}\sqrt{-g}\bar{M}_{kl}\,\times\\ &\times\left((n-1)A_{(n-1)}^{(k)\alpha\sigma_1\cdots\sigma_{n-2}}A^{(l)\beta}_{(n-1)\sigma_1\cdots\sigma_{n-2}}-\frac{1}{2}g^{\alpha\beta}A^{(k)}_{(n-1)\sigma_1\cdots\sigma_{n-1}}A_{(n-1)}^{(l)\sigma_1\cdots\sigma_{n-1}}\right)\\ &= -\frac{1}{2}e^\gamma\frac{\delta}{\delta g^{\alpha\beta}}(\bar{M}_{kl}A^{(k)}_{n-1}\wedge *A^{(l)}_{n-1}) = \frac{\delta\mathcal{L}'_b}{\delta g^{\alpha\beta}}\,,\end{aligned} \tag{4.12}$$

where we have defined $\bar{M}_{kl} = \mathcal{K}_{ij}M^i{}_kM^j{}_l = (M^T\mathcal{K}M)_{kl}$. The equivalence of the field equations for the Kaluza-Klein field φ are also easily checked.

As a result we have a new D-dimensional lagrangian which yields the D-dimensional field equations and also the constraint:

$$\begin{aligned}\mathcal{L}_D &= R*1-\frac{1}{2}d\varphi\wedge *d\varphi-\frac{1}{2}e^{-2(D-1)\alpha\varphi}\mathcal{F}_2\wedge *\mathcal{F}_2\,+\\ &+\frac{1}{2}\left(\Omega^{-1}M\right)_{ij}\left[(-1)^{n-1}A^i_{n-1}\wedge DA^j_{n-1}+e^\gamma\tilde{M}^j{}_kA^i_{n-1}\wedge *A^k_{n-1}\right]+\\ &+\frac{1}{4}\mathrm{tr}\left(\mathcal{D}\mathcal{K}\wedge *\mathcal{D}\mathcal{K}^{-1}\right)-\frac{1}{2}e^{2(D-1)\alpha\varphi}\,\mathrm{tr}\left(M^2+M\mathcal{K}^{-1}M^T\mathcal{K}\right)*1\,.\end{aligned} \tag{4.13}$$

If M is not invertible, there is a similar action with a Chern-Simons action for the massive $n-1$ form gauge fields, and a standard action for the massless gauge fields. First note that the lagrangian (4.8) can be written in flat indices as

$$\mathcal{L}'_b = \frac{1}{2}P_{ab}\left[(-1)^{n-1}A^a_{n-1}\wedge\tilde{\mathcal{D}}A^b_{n-1}+e^\gamma\tilde{M}^b{}_cA^a_{n-1}\wedge *A^c_{n-1}\right], \tag{4.14}$$

where $P_{ab} = P_{ij}(\mathcal{V}^{-1})^i{}_a(\mathcal{V}^{-1})^j{}_b = (\Omega^{-1})_{ac}M^c{}_b$ and $\tilde{M}^a{}_b = \tilde{M}^i{}_j\mathcal{V}^a_i(\mathcal{V}^{-1})^j{}_b$. Note that one has $P^{ab} = P_{cd}\Omega^{ca}\Omega^{db} = (-1)^{n-1}M^{ab}$. When M is not invertible, A^α_{n-1} drops out from this lagrangian, which is now just a lagrangian for $\bar{A}_{(n-1)\alpha'}$:

$$\mathcal{L}'_{b1} = \frac{1}{2}m^{\alpha'\beta'}\left[\bar{A}_{(n-1)\alpha'}\wedge\tilde{\mathcal{D}}\bar{A}_{(n-1)\beta'}+(-1)^{n-1}e^\gamma\delta_{\beta'\gamma'}m^{\gamma'\rho'}\bar{A}_{(n-1)\alpha'}\wedge *\bar{A}_{(n-1)\rho'}\right]. \tag{4.15}$$

Here we have used that $\tilde{M} = QM = \Omega\mathcal{K}M$ so that $\tilde{M}^a{}_b = \Omega^{ac}\delta_{cd}M^d{}_b$. Note that $m^{\alpha'\beta'} = (-1)^n m^{\beta'\alpha'}$ because of (2.32) and $m^{\alpha'\beta'}\delta_{\beta'\gamma'}m^{\gamma'\rho'}$ is always symmetric, as it should be. It is easy to see that the field equations of (4.15) for the gauge fields $\bar{A}_{(n-1)\alpha'}$ does indeed give the constraint (4.6). The lagrangian for A^α arises from (2.37) (with an extra factor of 1/2):

$$\mathcal{L}_{b2} = -\frac{1}{4}e^{-\gamma}\delta^{\alpha\beta}\bar{H}_{(n)\alpha}\wedge *\bar{H}_{(n)\beta} - \frac{1}{4}e^{\gamma}\delta_{\alpha\beta}\mathcal{D}A^\alpha_{n-2}\wedge *\mathcal{D}A^\beta_{n-2} \tag{4.16}$$

subject to the constraint (4.7), which can be used to eliminate either A^α_{n-2} or A^α_{n-1}. Choosing the first, the lagrangian for A^α_{n-1} is

$$\mathcal{L}'_{b2} = -\frac{1}{2}e^{\gamma}\delta^{\alpha\beta}\mathcal{D}\bar{A}_{(n-1)\alpha}\wedge *\mathcal{D}\bar{A}_{(n-1)\beta}\,. \tag{4.17}$$

Then the total lagrangian is

$$\mathcal{L}'_D = \mathcal{L}_g + \mathcal{L}_s + \mathcal{L}'_{b1} + \mathcal{L}'_{b2}\,, \tag{4.18}$$

where $\mathcal{L}_g$ and $\mathcal{L}_s$ are as in (2.23). It is straightforward to show that these give the right field equations, by an argument similar to that in the invertible case above.

4.3 $G = \mathrm{SL}(2,\mathbb{R})$ case

In this subsection we will consider the case $G = \mathrm{SL}(2,\mathbb{R})$. In this case the matrices $\mathcal{K}$ and Ω are as in (2.9) and (3.11). There are three distinct reductions corresponding to the three conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$ as discussed in section 2. The mass matrices representing the three conjugacy classes are given in (2.39). Now we will give the reduced lagrangians for each mass matrix M_e, M_h and M_p.

M_e**:** there are two massive, $(n-1)$-forms in the theory which we will call A^1 and A^2. This is an SO(2)-gauged theory since M_e generates the SO(2) subgroup of $\mathrm{SL}(2,\mathbb{R})$. (If $n=2$, there are additional gauge fields and the gauge group is $ISO(2)$.) This is the only case the theory has a stable minimum of the potential [14]. The global minimum of the potential is at $\chi = \phi = 0$. The lagrangian is:

$$\begin{aligned}\mathcal{L}_D = &\; R*1 - \frac{1}{2}d\varphi\wedge *d\varphi - \frac{1}{2}e^{-2(D-1)\alpha\varphi}\mathcal{F}_2\wedge *\mathcal{F}_2 + \\ &+ \frac{1}{2}m\{(-1)^{n-1}A^1\wedge DA^1 + (-1)^{n-1}A^2\wedge DA^2 - \\ &\qquad - me^{\gamma}e^{\phi}[A^1\wedge *A^1 + (e^{-2\phi}+\chi^2)A^2\wedge *A^2 + 2\chi A^1\wedge *A^2]\} + \\ &+ \frac{1}{4}\mathrm{tr}(\mathcal{D}\mathcal{K}\wedge *\mathcal{D}\mathcal{K}^{-1}) - 2e^{2(D-1)\alpha\varphi}m^2[\sinh^2\phi + \chi^2(2+e^{2\phi}(2+\chi^2))]*1\,.\end{aligned} \tag{4.19}$$

M_h**:** there are two massive, $(n-1)$-forms in the theory which we will call A^1 and A^2, as before. The gauge group is SO(1,1) in this case (for $n>2$). The lagrangian is:

$$\begin{aligned}\mathcal{L}_D = &\; R*1 - \frac{1}{2}d\varphi\wedge *d\varphi - \frac{1}{2}e^{-2(D-1)\alpha\varphi}\mathcal{F}_2\wedge *\mathcal{F}_2 + \\ &+ \frac{1}{2}m\{(-1)^{n-1}2A^1\wedge DA^2 - \\ &\qquad - me^{\gamma}e^{\phi}[\chi A^1\wedge *A^1 + \chi A^2\wedge *A^2 + (e^{-2\phi}+\chi^2+1)A^1\wedge *A^2]\} + \\ &+ \frac{1}{4}\mathrm{tr}(\mathcal{D}\mathcal{K}\wedge *\mathcal{D}\mathcal{K}^{-1}) - 2e^{2(D-1)\alpha\varphi}m^2[1+\chi^2e^{2\phi}]*1\,.\end{aligned} \tag{4.20}$$

M_p**:** there is one massive $(n-1)$-form field $\bar{A}_1$, one massless $(n-1)$-form field $\bar{A}_2$ and one massless $(n-2)$-form field B^2. However one can eliminate B^2 by using the reduced constraint (4.7), as was discussed in the previous subsection. The gauge group is $\mathrm{SO}(1,1)$ in this case (for $n>2$).

$$\begin{aligned}\mathcal{L}_D = &R*1 - \frac{1}{2}d\varphi\wedge *d\varphi - \frac{1}{2}e^{-2(D-1)\alpha\varphi}\mathcal{F}_2\wedge *\mathcal{F}_2 + \\ &+\frac{1}{2}m[\bar{A}_1\wedge \mathcal{D}\bar{A}_1 + (-1)^{n-1}e^{\gamma}m\bar{A}_1\wedge *\bar{A}_1] - \frac{1}{2}e^{\gamma}\mathcal{D}\bar{A}_2\wedge *\mathcal{D}\bar{A}_2 + \\ &+\frac{1}{4}\operatorname{tr}(\mathcal{D}\mathcal{K}\wedge *\mathcal{D}\mathcal{K}^{-1}) - \frac{1}{2}e^{2(D-1)\alpha\varphi}m^2(e^{-\phi}+e^{\phi}\chi^2)^2 * 1.\end{aligned}$$

5. Supergravity applications

In this section, we will apply our results to the twisted reduction of supergravity theories in $d=D+1=4,6,8$ dimensions to $D=3,5,7$. We will discuss general features here, and give details of the full lagrangians and of the classification of theories elsewhere.

5.1 Reduction of $d=8$ maximal supergravity

The $N=2$ $d=8$ maximal supergravity [2] can be obtained from 11-dimensional supergravity by toroidal compactification and has field equations invariant under the duality group $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(3,\mathbb{R})$. The bosonic fields consist of a metric, a 3-form gauge field A_3, 6 vector fields in the **(2,3)** representation of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(3,\mathbb{R})$, 3 2-form gauge fields in the **(1,3)** representation of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(3,\mathbb{R})$, and scalars taking values in the coset space $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)\times\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. The gauge field A_3 combines with the dual gauge field $\tilde{A}_3$ to form a doublet under $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(3,\mathbb{R})$ is a symmetry of the action whereas $\mathrm{SL}(2,\mathbb{R})$ is a symmetry of the field equations only, as it acts through electro-magnetic duality on the 3-form gauge fields.

There is a consistent truncation of this theory where only the $\mathrm{SL}(3,\mathbb{R})$ singlets are kept and all the other fields are set to zero [29]. Then the truncated theory consists of a metric, a 3-form gauge field and scalars taking values in $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, with an $\mathrm{SL}(2,\mathbb{R})$ S-duality symmetry. This truncated theory is precisely of the form (3.1) with $n=4$ and the twisted reduction with an $\mathrm{SL}(2,\mathbb{R})$ twist gives three distinct reduced theories corresponding to the three conjugacy classes, with lagrangians (4.19), (4.20) or (4.21).

This can be extended to the full theory, as the reduction of the fields that are not $\mathrm{SL}(3,\mathbb{R})$ singlets is a standard Scherk-Schwarz reduction. There are some complications resulting from the Chern-Simons interactions of the $d=8$ theory, and we will not present the full results here. There are three distinct classical theories, while the distinct quantum theories correspond to the distinct $\mathrm{SL}(2,\mathbb{Z})$ conjugacy classes.

5.2 Reduction of $d=4$, $N=4$ supergravity

$N=4$ supergravity coupled to p vector multiplets has an $O(6,p)$ symmetry of the action and an $\mathrm{SL}(2,\mathbb{R})$ S-duality symmetry of the equations of motion. The vector fields A_1^I $(I=1,2,\ldots,6+p)$ are in the fundamental **6+p** representation of $O(6,p)$ and combine with dual potentials $\tilde{A}_1^I$ to form $6+p$ doublets A_1^{mI} $(m=1,2)$ transforming in the **(2,6+p)** of $\mathrm{SL}(2,\mathbb{R})\times O(6,p)$. The scalars take values in the coset $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)\times O(6,22)/O(6)\times O(22)$. The scalars in $O(6,22)/O(6)\times O(22)$ can be represented by a coset space metric $\mathcal{N}_{IJ}$ while the 2 scalars ϕ,χ in $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$ can be represented by a coset space metric $\mathcal{K}_{mn}$ which is of the same form as (2.9).

The lagrangian for the bosonic sector can be written as [20, 30, 31]:

$$\begin{aligned}\mathcal{L} \,=\, & R*1+\frac{1}{4}\,\mathrm{tr}(d\mathcal{K}\wedge *d\mathcal{K}^{-1})+\frac{1}{4}\,\mathrm{tr}(d\mathcal{N}\wedge *d\mathcal{N}^{-1})-\frac{1}{2}e^{-\phi}F_2^I\mathcal{N}_{IJ}\wedge *F_2^J-\\ & -\frac{1}{2}\chi F_2^I L_{IJ}\wedge F_2^J\,,\end{aligned}\tag{5.1}$$

where L is the $O(6,p)$ invariant metric and the matrices $\mathcal{N}$ and L satisfy

$$\mathcal{N}^T=\mathcal{N}\,,\qquad \mathcal{N}^TL\mathcal{N}=L\,.\tag{5.2}$$

Now the vector field equation can be written as $dG_2^I=0$ where

$$G_2^I=(L^{-1})^{IJ}\frac{\delta\mathcal{L}}{\delta F_2^J}=-e^{-\phi}\mathcal{R}^I{}_J*F_2^J-\chi F_2^I\tag{5.3}$$

and the matrix $\mathcal{R}$ is defined as

$$L_{PI}\mathcal{R}^I{}_J=\mathcal{N}_{PJ}\,.\tag{5.4}$$

Note that $\mathcal{R}^2=1$. Now we can write

$$\begin{aligned}\mathcal{L}' \,=\, & R*1+\frac{1}{4}\,\mathrm{tr}(d\mathcal{K}\wedge *d\mathcal{K}^{-1})+\frac{1}{4}\,\mathrm{tr}(d\mathcal{N}\wedge *d\mathcal{N}^{-1})+\\ & +\frac{1}{2}F_2^IL_{IJ}\wedge G_2^J\,.\end{aligned}$$

As before, the field equations $dG_2^I=0$ imply the existence of dual potentials $\tilde{A}_1^I$, with $G_2^I=d\tilde{A}_1^I$. Then the full set of vector fields A_1^i in the doubled formalism is $A_1^{mI}=(A_1^I,\tilde{A}_1^I)$ where $i=1,\ldots,2(6+p)$ becomes the composite index mI. The field strengths are the $6+p$ $\mathrm{SL}(2,\mathbb{R})$-doublets:

$$H_2^I=\begin{pmatrix} dA_1^I\\ d\tilde{A}_1^I\end{pmatrix}.\tag{5.5}$$

We also impose the twisted self-duality constraint

$$H_2^{mI}=J^m{}_n\mathcal{R}^I{}_J*H_2^{nJ}\,,\tag{5.6}$$

where $J^m{}_n$ is as in (3.10), $J^m{}_n=\Omega^{mp}\mathcal{K}_{pn}$. So the matrix Q in (3.19) is now the $(12+2p)\times(12+2p)$ matrix

$$Q=J\otimes\mathcal{R}\tag{5.7}$$

which satisfies $Q^2=-1$ since $J^2=-1$ and $R^2=+1$. The doubled lagrangian

$$\begin{aligned}\mathcal{L} \,=\, & R*1+\frac{1}{4}\,\mathrm{tr}(d\mathcal{K}\wedge *d\mathcal{K}^{-1})+\frac{1}{4}\,\mathrm{tr}(d\mathcal{N}\wedge *d\mathcal{N}^{-1})-\\ & -\frac{1}{4}\mathcal{N}_{IJ}H_2^{mI}\mathcal{K}_{mn}\wedge *H_2^{nJ}\end{aligned}\tag{5.8}$$

gives the same field equations as those of (5.1) when the constraint equation (5.6) is imposed [21]. This lagrangian is of the same form as (3.8), with $\mathcal{K}_{ij}$ given by

$$\mathcal{K}_{mI\ nJ} = \mathcal{K}_{mn}\mathcal{N}_{IJ}\,.$$

Then the Scherk-Schwarz reduction of (5.8) with mass matrix $M^m{}_n$ in the Lie algebra of $\mathrm{SL}(2,\mathbb{R})$ can be performed as before and the three dimensional lagrangian that one obtains is:

$$\begin{aligned}\mathcal{L}_3' &= R*1 - \frac{1}{2}d\varphi\wedge *d\varphi - \frac{1}{2}e^{-2\varphi}\mathcal{F}_2\wedge *\mathcal{F}_2 + \frac{1}{8}\,\mathrm{tr}(d\mathcal{N}\wedge *d\mathcal{N}^{-1}) - \\ &\quad - \frac{1}{4}e^{-\varphi}\mathcal{N}_{IJ}H_2^{mI}\mathcal{K}_{mn}\wedge *H_{n2}^{J} - \frac{1}{4}e^{\varphi}\mathcal{N}_{IJ}H_1^{mI}\mathcal{K}_{mn}\wedge *H_1^{nJ} + \\ &\quad + \frac{1}{4}\mathrm{tr}(\mathcal{D}\mathcal{K}\wedge *\mathcal{D}\mathcal{K}^{-1}) - \frac{1}{2}e^{2\varphi}\mathrm{tr}(M^2 + M\mathcal{K}^{-1}M^T\mathcal{K}) * 1\,. \end{aligned} \tag{5.9}$$

This lagrangian is to be supplemented by the reduced constraint

$$H_2^{mI} = e^{\varphi}J^m{}_n\mathcal{R}^I{}_J * H_1^{nJ}\,. \tag{5.10}$$

When M is invertible, this becomes

$$DA_1^{mI} = e^{\varphi}J^m{}_n\mathcal{R}^I{}_J M^n{}_p * A_1^{pJ}\,, \tag{5.11}$$

after gauging the Stuckelberg fields away, as in section 2. As before one can find a three dimensional lagrangian from which the field equations and the constraint can be derived. This lagrangian is (for invertible M):

$$\begin{aligned}\mathcal{L}_3 &= R*1 - \frac{1}{2}d\varphi\wedge *d\varphi - \frac{1}{2}e^{-2\varphi}\mathcal{F}_2\wedge *\mathcal{F}_2 + \frac{1}{4}\,\mathrm{tr}(d\mathcal{N}\wedge *d\mathcal{N}^{-1}) + \\ &\quad + \frac{1}{2}(\Omega^{-1}M)_{mn}(-L_{IJ}A_1^{mI}\wedge DA_1^{nJ} + \mathcal{N}_{IJ}e^{\varphi}\tilde{M}^n{}_pA_1^{mI}\wedge *A_1^{pJ}) + \\ &\quad + \frac{1}{4}\mathrm{tr}(\mathcal{D}\mathcal{K}\wedge *\mathcal{D}\mathcal{K}^{-1}) - \frac{1}{2}e^{2\varphi}\mathrm{tr}(M^2 + M\mathcal{K}^{-1}M^T\mathcal{K}) * 1\,. \end{aligned} \tag{5.12}$$

There is a similar action for the case in which M is non-invertible.

5.3 Reduction of $d = 4$, $N = 8$ supergravity

The $D = 4, N = 8$ theory has E_7 duality symmetry of the equations of motion. There are 70 scalars taking values in the coset $E_7/SU(8)$, and 28 vector fields A^I which combine with their duals to give A^i transforming as a **56** of E_7. The bosonic action can be written as (3.8) with the constraint (3.19) where Q is as in (3.20) and Ω^{ij} is the symplectic invariant of E_7 [22]. Now $\mathcal{K}$ is the matrix which parametrizes the scalar coset $E_7/SU(8)$. The theory can be reduced to 3-dimensions using any mass matrix M in the Lie algebra of E_7. Naively, this introduces 133 mass parameters, but these theories are not all independent and the independent theories correspond to the distinct conjugacy classes; the classification of conjugacy classes in this case is not known. The matrix $M^{ab} = M^a{}_c\Omega^{cb}$ introduced in section 2 is a symmetric matrix since $n = 2$ in (2.32) so, by choosing a suitable basis, it can be brought into the diagonal form:

$$M^{ab} = \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_{56} \end{pmatrix} . \tag{5.13}$$

For example, consider performing the Scherk-Schwarz reduction with the Lie algebra element M^{ab} of the form:

$$M^{ab} = m \begin{pmatrix} 0_l & & 0 \\ & 1_p & \\ 0 & & -1_q \end{pmatrix} , \tag{5.14}$$

where $l + p + q = 56$. Then one obtains a 3-dimensional theory with one mass parameter with p massive, self-dual vector fields, q massive, anti-self-dual vector fields and l massless vector fields which are dual to the l massless scalar fields coming from the reduction of the vector field in the 4-dimensional theory.

5.4 Reduction of $d = 6$ supergravity

The $d = 6$ theory of [32], obtained from a truncation of the toroidal compactification of IIB supergravity, has an

$$\mathrm{SO}(2,2) \equiv \mathrm{SL}(2,\mathbb{R})_{\mathrm{EM}} \times \mathrm{SL}(2,\mathbb{R})_{\mathrm{IIB}} \tag{5.15}$$

symmetry of the equations of motion. The $\mathrm{SL}(2,\mathbb{R})_{IIB}$ is inherited from the $\mathrm{SL}(2,\mathbb{R})$ symmetry of IIB in ten dimensions and is a symmetry of the action in the six dimensional theory. However $\mathrm{SL}(2,\mathbb{R})_{EM}$ is a symmetry of the field equations only. The bosonic lagrangian is:

$$\begin{aligned} \mathcal{L} = R * 1 + \frac{1}{4}\operatorname{tr}(d\mathcal{K} \wedge *d\mathcal{K}^{-1}) + \frac{1}{4}\operatorname{tr}(d\mathcal{N} \wedge *d\mathcal{N}^{-1}) - \\ - \frac{1}{2}e^{-\phi_1} F^I \mathcal{N}_{IJ} \wedge *F^J - \frac{1}{2}\chi_1 F^I \Omega_{IJ} \wedge F^J . \end{aligned} \tag{5.16}$$

Here $F^I = dA_2^I$ are the two 3-form field strengths and $I, J = 1, 2$ are $\mathrm{SL}(2,\mathbb{R})_{IIB}$ indices. We also introduce $\mathrm{SL}(2,\mathbb{R})_{EM}$ indices $m, n = 1, 2$. There are two $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$ scalar cosets in the theory. $e^{-\phi_1}$ and χ_1 parametrize the scalar coset $\mathrm{SL}(2,\mathbb{R})_{EM}/\,\mathrm{SO}(2)$, represented by the matrix $\mathcal{K}_{mn}$. The other two scalars $e^{-\phi_2}$ and χ_2, parametrize the scalar coset $\mathrm{SL}(2,\mathbb{R})_{IIB}/\,\mathrm{SO}(2)$, which is represented by the matrix $\mathcal{N}_{IJ}$. The invariant matrices are Ω^{mn}, Ω^{IJ}.

The lagrangian (5.16) is of the same form as (5.1), where now I ranges from 1 to 2 and the $O(6,p)$ invariant L_{IJ} has been replaced by the $\mathrm{SL}(2,\mathbb{R})_{IIB}$ invariant matrix Ω_{IJ}. So (5.16) is equivalent to the doubled lagrangian (5.8) (now with 3-form field strengths H_3) when supplemented by the constraint (5.6). Note that the matrix Q in (5.7) now satisfies $Q^2 = +1$, as it should in 6 dimensions, since now $\mathcal{R}^2 = -1$, whereas $\mathcal{R}^2 = +1$ and hence $Q^2 = -1$ in the 4-dimensional case.

By performing the Scherk-Schwarz reduction of the doubled lagrangian with monodromy in $\mathrm{SL}(2,\mathbb{R})_{EM}$, one obtains the following auxiliary five-dimensional lagrangian:

$$\begin{aligned}\mathcal{L}'_5 &= R*1-\frac{1}{2}d\varphi\wedge *d\varphi-\frac{1}{2}e^{-4/\sqrt{6}\varphi}\mathcal{F}_2\wedge *\mathcal{F}_2+\frac{1}{4}\,\mathrm{tr}(d\mathcal{N}\wedge *d\mathcal{N}^{-1})-\\ &-\frac{1}{4}e^{-2/\sqrt{6}\varphi}\mathcal{N}_{IJ}H_3^{mI}\mathcal{K}_{mn}\wedge *H_3^{nJ}-\frac{1}{4}e^{2/\sqrt{6}\varphi}\mathcal{N}_{IJ}H_2^{mI}\mathcal{K}_{mn}\wedge *H_2^{nJ}+\\ &+\frac{1}{4}\mathrm{tr}(\mathcal{D}\mathcal{K}\wedge *\mathcal{D}\mathcal{K}^{-1})-\frac{1}{2}e^{4/\sqrt{6}\varphi}\,\mathrm{tr}(M^2+M\mathcal{K}^{-1}M^T\mathcal{K})*1\,. \end{aligned}\tag{5.17}$$

This is to be supplemented by the five-dimensional reduced constraint

$$H_3^{mI}=e^{2/\sqrt{6}\varphi}J^m{}_n\mathcal{R}^I{}_J*H_2^{nJ}\,. \tag{5.18}$$

When M is invertible one can gauge the Stuckelberg fields away and in this gauge the constraint in (5.18) takes the form

$$DA_2^{mI}=-e^{2/\sqrt{6}\varphi}J^m{}_n\mathcal{R}^I{}_JM^n_{\ p}*A_2^{pJ}\,. \tag{5.19}$$

The five dimensional reduced lagrangian from which the reduced constraint (5.19) and the field equations of (5.17) can be derived is obtained by using the techniques of the previous sections:

$$\begin{aligned}\mathcal{L}_5 &= R*1-\frac{1}{2}d\varphi\wedge *d\varphi-\frac{1}{2}e^{-4/\sqrt{6}\varphi}\mathcal{F}_2\wedge *\mathcal{F}_2+\frac{1}{4}\,\mathrm{tr}(d\mathcal{N}\wedge *d\mathcal{N}^{-1})+\\ &+\frac{1}{2}(\Omega^{-1}M)_{mn}(\Omega_{IJ}A_2^{mI}\wedge DA_2^{nJ}+\mathcal{N}_{IJ}e^{2/\sqrt{6}\varphi}\tilde{M}^n_{\ p}A_2^{nI}\wedge *A_2^{pJ})+\\ &+\frac{1}{4}\mathrm{tr}(\mathcal{D}\mathcal{K}\wedge *\mathcal{D}\mathcal{K}^{-1})-\frac{1}{2}e^{4/\sqrt{6}\varphi}\,\mathrm{tr}(M^2+M\mathcal{K}^{-1}M^T\mathcal{K})*1\,. \end{aligned}\tag{5.20}$$

5.5 Reduction of $d=6$, $N=8$ supergravity

The maximal supergravity in six dimensions has noncompact global symmetry $\mathrm{SO}(5,5)$, which can be realized at the level of field equations only [24]. There are five 3-form field strengths which split into five self-dual ones and five anti-self dual ones, and these ten transform as a **10** of $\mathrm{SO}(5,5)$. There are 25 scalar fields in the theory and they parametrize the coset space $\mathrm{SO}(5,5)/\mathrm{SO}(5)\times\mathrm{SO}(5)$. The bosonic lagrangian can be written as (3.8), plus terms which we will not give explicitly here involving the vector fields, with the constraint (3.19) where Q is as in (3.20) and Ω^{ij} is the symplectic invariant of $\mathrm{SO}(5,5)$ [21]. Now $\mathcal{K}$ is the matrix which parametrizes the scalar coset $\mathrm{SO}(5,5)/\mathrm{SO}(5)\times\mathrm{SO}(5)$. The theory can be reduced to 5-dimensions using any mass matrix M in the Lie algebra of $\mathrm{SO}(5,5)$. The number of distinct reductions is given by the number of conjugacy classes of $\mathrm{SO}(5,5)$.

Consider the matrix $M^{ab}=M^a_{\ c}\Omega^{cb}$ introduced in section 2. It is an anti-symmetric matrix since $n=3$ in (2.32). So in a particular basis it can be brought into the skew-diagonal form:

$$M^{ab}=\begin{pmatrix} 0 & m_1 & & & \\ -m_1 & 0 & & 0 & \\ & & \ddots & & \\ & 0 & & 0 & m_5 \\ & & & -m_5 & 0\end{pmatrix}. \tag{5.21}$$

Consider a mass matrix of the form

$$M^{ab} = m \begin{pmatrix} \mathbf{0}_l & & & & & & \\ & 0 & 1 & & & & \\ & -1 & 0 & & & & \\ & & & \ddots & & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}, \tag{5.22}$$

where there are l zero eigenvalues and the number of skew-diagonal blocks is $(10 - l)/2$. On reduction, one obtains, in five dimensions, a gauged theory with one mass parameter including $10 - l$ massive self-dual 2-form fields and l massless 2-form fields, which could be dualised to l massless 1-form fields.

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